ORDINARY DIFFERENTIAL GRÖBNER BASES

Volker Weispfenning
(Universität Passau)

The talk reviews the current state of research concerning Gröbner bases for ordinary differential polynomials in several differential variables. The main topic will be the parallels and differences to algebraic Gröbner basis theory for multivariate polynomials. This concerns both the Gröbner basis theory itself and its applications, in particular its relevance for the solvability of systems of ODEs. We will also investigate the existence of parametric differential Gröbner bases, i.e. possible analogues of algebraic comprehensive Gröbner bases.

A FEW OF MY FAVORITE UNIVERSALITY THEOREMS

Shmuel Onn
(Technion - Israel Institute of Technology)

I will describe several universality results in Gröbner bases theory and its applications. First, in algebraic statistics, where Gröbner bases enable a random walk on the space of contingency tables with fixed released margins, I'll show that a simple class of table models indexed by a single parameter is universal and effectively hosts any toric ideal. Second, I'll exhibit efficiently constructible polyhedra which are universal for the Hilbert scheme of zero dimensional ideals and enable blowing any Gröbner basis to a universal one in polynomial time. Third, applying commutative algebra to combinatorial optimization, I'll demonstrate that normal form computations over very simple ideals are universal for the class number-P, and capture any counting problem.

BOOLEAN GRÖBNER BASES ARE ALWAYS COMPREHENSIVE

Yosuke Sato
(Tokyo University of Science)

A commutative ring with identity is called a boolean ring if each element is idempotent. For a boolean ring **B**, a quatient ring $B[X_1, \ldots, X_n]/\langle X_1^2 - X_1, \ldots, X_n^2 - X_n\rangle$ is called a boolean polynomial ring. A Gröbner basis in a boolean polynomial ring is called a boolean Gröbner basis. We show that any boolean Gröbner basis w.r.t. a cirtain term order is always strongly stable. This result enable us to compute comprehensive Gröbner bases in boolean polynomial rings with minimum computation costs. Out result also gives alternative ideal theoretic proofs for several classical theorems of boolean algebra such as an extention theorem or a Nullstellensatz.

ALGORITHMS FOR LOCAL D-MODULES

Nobuki Takayama
(Kobe University)

Let us consider 3 rings

$$\begin{aligned} \mathbf{K}[x] &= \mathbf{K}[x_1, \ldots, x_n], \\ \mathcal{O}_0^{alg} &= \{f/g \mid f, g \in \mathbf{K}[x], g(0) \neq 0\}, \\ \hat{\mathcal{O}}_0^{an} &= \mathbf{K}[[x_1, \ldots, x_n]] \end{aligned}$$

We are interested in algorithmic study of ideals or modules over these rings and analogous rings of differential operators.

For the first ring — the ring of polynomials, the Buchberger algorithm plays the central role. For the second and the third ring, which we call the local case, the tangent cone algorithm and the Buchberger algorithm are fundamental.

In the last 10 years, algorithmic study of modules over the Weyl algebra $D = \mathbf{K}[x]\langle \partial_1, \ldots, \partial_n \rangle$ has been intensively done. What is known and what should be studied in the local case of the ring of differential operators? In this survey talk, we will discuss about the following topics.

1. The membership problem and the tangent cone algorithm.
2. Characteristic variety and flatness.
3. b-function and indicial polynomials.
4. Restriction and integration functors.
5. Resolutions and local solutions.
6. Gröbner fan.

MODULAR COMPUTATION OF GRÖBNER BASES AND ITS APPLICATION TO DYNAMIC EVALUATION

Masayuki Noro
(Kobe University)

When we compute Gröbner bases, it often takes long time because of computational difficulties in the ground field. For example, a polynomial GCD computation over Q often shows a rapid increase of coefficients in the intermediate polynomials during the execution of the Euclid algorithm. In GCD computation, a computation over a finite field (hereafter we call it modular computation) and the Hensel lifting can be applied and we can obtain the result with the computational cost according to the size of the result. In general Gröbner basis computations, various applications of modular computations have been proposed and a certain amount of their benefits is observed. In this talk we briefly review some of them and we show new applications of modular computation to the Gröbner basis computation over algebraic number fields and dynamic evaluation (DE).

The Gröbner basis computation of an ideal over an algebraic number field can be trivially done if we simply apply arithmetic operations of algebraic numbers. However it causes a kind of coefficient swells because the algebraic numbers usually are expressed as polynomials over Q. The same computation can be done as that over Q, by adding the defining polynomials of the ground fields to the input ideal. In this case the bottle neck is the computation of the reciprocal in the ground field and we show that we can apply a modular computation to the reciprocal computation.

DE requires a square-free polynomial as the defining polynomial of newly added algebraic numbers (exactly speaking a radical ideal as the defining ideal of the extended ground ring). In general the ground ring may have zero divisors and the ring can be decomposed when we encounter a zero divisor. This is practical because the expensive polynomial factorization is not necessary. However, in the existing method, the detection of an element being a zero divisor is done simply by applying the Euclid algorithm, and its computational cost is the worst when the element is a unit. In this talk we reduce the detection of a zero divisor to a modular computation of an ideal quotient. Furthermore, the complementary component in he decomposition of the ground ring is obtained by the Chinese remainder computation of a Gröbner basis. In general it is difficult to check the validity of the result of Chinese remainder computation. In our case we can check it by using simple a linear algebra argument.

AUTOMATIC SELECTION OF TERM ORDERINGS FOR EFFICIENT COMPUTATION OF GRÖBNER BASES

Hiroyuki Sawada
(National Institute of Advanced Industrial Science and Technology)

It is well known that a term order has a great influence on computational efficiency of Gröbner bases. In my presentation, I propose a new method of selecting a term order that enables efficient computation of Gröbner bases. The advantages of this method have been shown through experiments. The results of experiments also have illustrated that the degree-reverse-lexicographic ordering is not always good for computational efficiency, and that, even in the case of degree-reverse-lexicographic ordering, the ranking of variables has a significant effect on efficiency of Gröbner basis computation. Since this method helps users without sufficient knowledge about the computational algorithms in computing Gröbner bases efficiently, it can create a new application area of Gröbner bases including engineering.

On polynomial curves in the affine plane

Mitsushi Fujimoto
(Fukuoka University of Education)
Masakazu Suzuki
(Kyushu University)
Kazuhiro Yokoyama
(Rikkyo University)

A curve that can be parametrized by polynomials is called a polynomial curve. It is well-known that a polynomial curve has only one place at infinity. In 1977, Sathaye indicated the Abhyankar's question for curves with one place at infinity. Let C be a curve with one place at infinity, and Ω be the semigroup generated by pole orders of C at infinity. Is there a polynomial curve associated with Ω? In 1990, Sathaye-Stenerson suggested a candidate of counter example for this question. However, they could not give the answer to the question since the root computation for a huge polynomial system was required. We found a counter example for the Abhyankar's question using Gröbner basis computation. In this talk, we give the details.

COMPUTING GRÖBNER FANS AND TROPICAL VARIETIES

Rekha Thomas
(University of Washington)

The Gröbner fan of a polynomial ideal is a polyhedral fan whose cells index the distinct reduced Gröbner bases of the ideal. This complex carries a wealth of information beyond the list of reduced Gröbner bases of the ideal and can be used as a theoretical tool in research. In this talk I will describe algorithms for computing Gröbner fans of an arbitrary ideal. These algorithms have been implemented in the software package Gfan written by Anders Jensen. Using the package Jensen has constructed an ideal in four variables whose Gröbner fan is not the normal fan of any polyhedron contrasting the well-known results of Bayer and Morrison on the regularity of the Gröbner fan of homogeneous ideals. Despite this result, Gröbner fans admit an acyclic orientation allowing them to be enumerated by reverse search. Gfan includes several other functionalities including algorithms for computing tropical varieties of ideals. I will explain these procedures as well.

Joint work with Tristram Bogart, Komei Fukuda, Anders Jensen, David Speyer and Bernd Sturmfels.

NEW ALGORITHMS & COMPLEXITY RESULTS ON THE COMPUTATION OF GRÖBNER BASES OF TORIC IDEALS

Jesus A. De Loera
(University of California, Davis)

A large variety of problems in Computer algebra, Combinatorics, Statistics, and Optimization can be formulated in the language of toric ideals. Examples of these are: computing Hilbert functions of monomial rings, computing Ehrhart polynomials and volumes of polytopes, sampling contingency tables, Solving hard integer programming problems, etc. For this reason Gröbner bases of toric ideals have particular importance and therefore fast algorithms are desirable. In this talk we present new algorithmic tools:

Our first result states that, using Barvinok's rational functions and for fixed number of variables and term order, the reduced Gröbner basis of a toric ideal can be computed in polynomial time. For computing the normal form of a monomial this can also be done in polynomial time. Our initial experiments suggest that in practice this approach, which entirely avoids S-pairs and Buchberger's algorithm, has potential for practical use.

For our second result we have also investigated the structure and complexity of toric ideals of 0/1 matrices. We have proved that any Gröbner bases of toric ideal is identical to the toric ideal of some special family of 0/1 matrices.

This reports on joint work with subsets of the following people Raymond Hemmecke, Peter Huggins, Dave Haws, Shmuel Onn, Bernd Sturmfels, and Ruriko Yoshida.

Gröbner techniques in the McKay correspondence

Diane Maclagan
(Rutgers University)

The McKay correspondence describes a connection between the representation theory of a finite subgroup G of $SL(2,C)$ and the geometry of the minimal resolution of singularities of C^2/G. This has focused attention on crepant resolutions of C^n/G for finite G contained in $GL(n,C)$, with a candidate given by Nakamura's G-Hilbert scheme and the related moduli of representations of the McKay quiver. When G is abelian these spaces are toric, and have completely concrete descriptions using the Gröbner fan of a related module. This allows us to do specific computations, and find some pathological examples. This is joint work with Alastair Craw and Rekha Thomas.

On the Gröbner walks

Yasuko Matsui
(Tokai University)

The Gröbner walk is a method for converting a given Gröbner basis of a polynomial ideal I to a Gröbner basis of I with respect to another term order and proposed by Collart, Kalkbrener, and Mall. It is to trace the line between vectors in different Gröbner cones. Amrhein, Gloor, and Küchlin presented an application of the Gröbner walk in 1997. Moreover, they reported empirical results from system solving, including comparisions with Buchberger's algorithm and the FGLM basis conversion method. In 2000, Trans proposed the improved Gröbner walk method and showed the performance of their method in comparision with other known methods. Recently, Fukuda, Jensen, Lauritzen, and Thomas proposed the generic Gröbner walk method. The key technique is known as the lexicographic perturbation method used in optimization and computational geometry, and used by many reliable implementations of the simplex method for linear programming finite. More precisely, they gave a new lifting step using reduction modulo the known Gröbner basis.

In this talk, we will survey some walk algorithms for computing Gröbner bases.

MARKOV BASES FOR SAMPLING CONTINGENCY TABLES WITH FIXED MARGINALS

Akimichi Takemura and Satoshi Aoki
(University of Tokyo)

Consider conditional tests for contingency tables with fixed marginals. If the size of table is large compared to the sample size n, n may be large enough to make enumeration of the sample space infeasible, but at the same time n may be not large enough to warrant the large sample χ^2 approximation. In this case Monte Carlo simulation of the P-value is desirable. Furthermore if the direct sampling from the conditional distribution is difficult (e.g. the test of no three-factor interactions in three-way tables), the MCMC method is a useful tool to employ.

By using Metropolis-Hastings method, once a connected Markov chain is constructed over the sample space, it can be changed to have a target distribution as the stationary distribution. Therefore the basic mathematical problem is the construction of a connected Markov chain. A fundamental result on this construction was given by Diaconis and Sturmfels (*Annals of Statistics*, 1998). They defined the notion of Markov basis for constructing a connected chain and showed that the Markov basis corresponds to a set of generators of toric ideals and hence can be obtained by Gröbner basis computation. Therefore Markov basis can be computed by computational algebra packages such as "4ti2" (Hemmecke and Hemmecke, 2003). Although rapid progresses are being made in algebraic algorithms, we find that these algorithms still have problems in computational time and in redundant outputs.

In a series of papers by Aoki and Takemura, we looked at the problem of the redundancy in the reduced Gröbner basis and obtained some results on minimal Markov basis and its uniqueness, without relying on Gröbner basis technology. In this talk, we summarize some of our findings.

RECENT DEVELOPMENT OF CODING THEORY RELATED TO GRÖBNER BASIS

Shojiro Sakata
(University of Electro-Communications)

In this talk we discuss recent development of algebraic coding theory and its interplay with Gröbner basis theory and algorithms. Their close connection was born in the trend from polynomials to ideals and from 1-D codes to multi-D codes. Fast decoding algorithms have been a target of the most active investigations in coding theory, where the interconnection of ideal versus variety has been a key trick. Particularly, fast decoding and Gröbner basis have interacted with each other via multi-D linear recurrences. In the world of coding theory Gröbner basis theory and algorithms have played a very important role.

A CONNECTEDNESS RESULT IN POSITIVE CHARACTERISTIC

Uli Walther
(Purdue University)

This is a joint work with A. Singh (Utah).

How many components does a given projective variety have? This is easy to check over a small field like $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ due to methods for primary decomposition. Over $\mathbb{C}$, de Rham cohomology methods can be used. Let $\mathbb{K}$ be algebraically closed of positive characteristic. We show that the number of components of $\mathrm{Proj}(\mathbb{C}[\mathbf{x}]/I)$ is $\dim_{\mathbb{K}}(H^1_{st})$ where H^1_{st} is the stable image under the iterated Frobenius on $H^1_{\mathbf{x}}(\mathbb{C}[\mathbf{x}]/I)$. In the process we recover some results of Hartshorne–Speiser and Lyubeznik. We describe how the stable image H^1_{st} may be computed with a computer algebra system such as *Macaulay2*.

COMPUTING THE GLOBAL BRIESKORN LATTICE OF TAME POLYNOMIAL FUNCTIONS

Mathias Schulze
(Purdue University)

We consider a so-called cohomologically tame polynomial function. It has only isolated critical points and a certain good behaviour at infinity. We present an algorithm to compute its Gauss-Manin system, which is the D-module direct image of the structure sheaf. The key for our approach is the so-called Brieskorn lattice, the image of the top differential forms in the Gauss-Manin system. It is not a D-module but equipped with a differential structure given by the variable t=f and the inverse s of the partial derivative with respect to t. We explain how to compute this C[t,s]-module structure of the Brieskorn lattice, more precisely, a C[s]-basis and a matrix of the differential operator t. The main problem is that a priory we do not have a finite C[s]-presentation of the Brieskorn lattice. Our methods are based on Gröbner basis ideas: mainly we perform Gröbner basis computations compatible with certain natural filtrations. But termination and correctness of our algorithm use a deep result due to C. Sabbah: the Brieskorn lattice induces the Hodge filtration of a natural mixed Hodge structure. Several consequences of this fact are essential for our approach. Finally we can compute a so-called good basis of the Brieskorn lattice: the C[s]-matrix of t lives in degree 0 and 1 only. From these two constant matrices one can read off directly the spectral pairs or mixed Hodge numbers and the complex monodromy at infinity of the tame polynomial.

INVARIANTS OF SEMIQUASIHOMOGENEOUS ISOLATED SINGULARITIES

Yayoi Nakamura
(Kinki University)

Numerical invariants of a hypersurface isolated singularity will be defined as the multiplicity of holonomic systems attached to the singularity. We study a relation between the invariants and classical invariants. Especially, we consider the case of semiquasihomogeneous isolated singularities with inner modality smaller than or equal to four.

We have studied algebraic local cohomology classes which constitute, via the Grothendieck local residue pairing, the dual vector space to the Milnor algebra of a hypersurface isolated singularity. We have considered, in particular, an algebraic local cohomology class which generates the dual space in the context of D-modules. We showed that the hypersurface isolated singularity is quasihomogeneous if and only if the multiplicity of the associated first order holonomic system is equal to one. We have introduced a series of numerical invariants $\mu_f^{(k)}$ $(k = 1, 2, \dots)$ as the multiplicity of the k-th order holonomic system associated to the singularity.

We have noticed through these studies, that the multiplicity $\mu_f^{(1)}$ seemed to be related to Milnor number and Tjurina number of the singularity. In fact, we obtained an explicit formula between these invariants for semiquasihomogeneous unimodal and bimodal isolated singularity cases.

In order to examine whether the formula holds or not for general isolated singularities, we study semiquasihomogeneous hypersuface whose inner modality is smaller or equal to four.

LOCALIZATION OF HOLONOMIC D-MODULES: SEVERAL POINTS OF VIEW

Anton Leykin
(University of Illinois at Chicago)

In the first part of this talk we overview selected applications of the Gröbner bases techniques in the Weyl algebra. In particular, we recall algorithms due to Oaku, Takayama, and Walther that compute the localization of a holonomic D-module and its local cohomology.

Characteristic cycles (CCs) provide another tool to examine D-modules; CCs may be computed using the D-approach as well. However, given a CC of a regular holonomic D-module, it is also possible to give an algorithm (joint work with Josep Alvarez) that determines the CC of its localization using only computations in (commutative) polynomial rings.

Demonstrations of computations in the D-modules package for Macaulay 2 (joint work with Harry Tsai) accompany the presentation.

An algorithm for computing Noetherian differential operators

Shinichi Tajima
(Niigata University)

We consider Noetherian differential operators attached to a zero-dimensional primary ideal in a context of algebraic analysis. We use the theory of holonomic D-modules and the theory of algebraic local cohomology to show that Gröbner duality is completely equivalent to Grothendieck duality for a zero-dimensional case. We describe an effective algorithm for computing Noetherian differential operators and we discuss some applications.

Greedy Fans: A geometric approach to dual greedy algorithms

Hiroshi Hirai
(Kyoto University)

We study greedily solvable linear programs in a geometric way. Such linear programs have recently been considered by Faigle and Kern, and Kruger for antichains of posets, and by Frank for a class of lattice polyhedra, and by Kashiwabara and Okamoto for extreme points of abstract convex geometries. Our guiding principle is that solving linear programs is equivalent to finding a normal cone of a polyhedron which contains a given vector. Motivated by this observation, we introduce and investigate a class of simplicial fans, called *greedy fans*, whose membership problem can be greedily solvable. Relationship to Gröbner bases of toric ideal will be discussed.

ALGORITHMS OF COMPUTING LOCAL b FUNCTION BY USING GRÖBNER BASIS AND DIVISION IN $\widehat{\mathcal{D}}[s]$

Hiromasa Nakayama
(Kobe University)

For any polynomial $f \in C[x_1, \cdots, x_n]$, there uniquely exists the monic minimal degree polynomial $b(s)$ satisfying

$$\exists P(s) \in \widehat{\mathcal{D}}[s] \ s.t. \ P(s) \cdot f^{s+1} = b(s) f^s.$$

This polynomial is called the local b function of f. Oaku gave an algorithm of computing local b function by using Gröbner basis in D and saturation.[Oaku, T.: An algorithm of computing b function [1997])

I will give 2 new algorithms of computing local b function by using Gröbner basis and division algorithms in $\mathcal{D}[s]$.

One is an algorithm by using the Mora division algorithm in $\widehat{\mathcal{D}}[s]$ (Oaku, T., Takayama, N., Granger, M. :Tangent cone algorithm for homogenized differential operators [2003]). In $\widehat{\mathcal{D}}[s]$, we can solve ideal membership problem by the Mora division as in the case of the ring of polynomials. By utilizing this, we compute local b functions by an exhaustive search.

The second one is an algorithm utilizing approximate division algorithm in $\widehat{D}[s]$. Our approximate division algorithm returns correct quotient and remainder up to a given total degree. The remainder (or normal form) by Gröbner basis is unique. By utilizing these, we compute local b functions. This algorithm is analogous to an efficient algorithm computing global b function by Noro. (Noro, M. : An efficient modular algorithm for computing the global b function [2002]).

DIMENSION FORMULA OF SOLUTION SPACES OF $\mathcal{A}$-HYPERGEOMETRIC DIFFERENTIAL-DIFFERENCE SYSTEMS

Katsuyoshi Ohara (Kanazawa University)
Nobuki Takayama (Kobe University)

Let $A = (a_{ij})$ be a matrix in $M(d, n, \mathbf{Z}_{\geq 0})$. We suppose that the set of the column vectors of A spans $\mathbf{Z}^d$. We denote by $S_i : f(s_i) \mapsto f(s_i - 1)$ the difference operator with respect to a variable s_i.

Definition 1 *We call the following differential-difference system* $\mathbf{H}_A$ *an* $\mathcal{A}$*-hypergeometric differential-difference system:*

$$\begin{aligned} \left(\sum_{j=1}^{n} a_{ij} x_j \partial_j - s_i \right) \bullet f &= 0, \qquad (i = 1, \ldots, d) \\ \left(\partial_j - \prod_{i=1}^{d} S_i^{a_{ij}} \right) \bullet f &= 0. \qquad (j = 1, \ldots, n) \end{aligned}$$

Definition 2 *Let* I *be a left ideal of the ring of differential-difference operators*

$$D = \mathbf{C}(x_1, \ldots, x_n, s_1, \ldots, s_d)\langle \partial_1, \ldots, \partial_n, S_1, \ldots, S_d \rangle.$$

We define the rank of I *by*

$$\mathrm{rank}(I) = \dim_{\mathbf{C}(x,s)} D/I,$$

where D/I *is a vector space over* $\mathbf{C}(x, s)$*.*

Theorem 1 *The rank of* $\mathbf{H}_A$ *agrees with the normalized volume of* A*.*

Let a_i be the i-th column vector of the matrix A and $F(\beta, x)$ the integral

$$F(\beta, x) = \int_C \exp\left(\sum_{i=1}^{n} x_i t^{a_i} \right) t^{-\beta - 1} dt, \qquad t = (t_1, \ldots, t_d).$$

The integral $F(\beta, x)$ satisfies an $\mathcal{A}$-hypergeometric differential system "formally". Moreover, $F(s, x)$ satisfies an $\mathcal{A}$-hypergeometric differential-difference system "formally". We use the word "formally" because, there is no general and rigorous description about the cycle C. However, the integral representation gives an intuitive figure of what are solutions of $\mathcal{A}$-hypergeometric differential-difference system.

Rank theories of $\mathcal{A}$-hypergeometric differential system have been developed since Gel'fand, Zelevinsky and Kapranov. In the end of 1980's, under the condition that the points lie on a same hyperplane, they proved that the rank of $\mathcal{A}$-hypergeometric differential system $H_A(\beta)$ agrees with the normalized volume of A if the toric ideal

I_A has the Cohen-Macaulay property. After their result had been gotten, many people proved theorems for equivalence of the rank and the volume under various conditions. In particular, Matusevich, Miller and Walther proved that I_A has the Cohen-Macaulay property if there exists a parameter β such that the rank of $H_A(\beta)$ is greater than the volume of A. ([2])

We proved Theorem 1 utilizing theorems above, uniform convergence of series solutions, and Mutsumi Saito's results for contiguity relations. Finally, we note that, for studying $\mathcal{A}$-hypergeometric differential-difference system, we wrote a program "yang" ([3], [4]) on a computer algebra system Risa/Asir and did experimentations on computers.

Example 1 *Put* $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. *Then the rank of* $\mathcal{A}$*-hypergeometric differential-difference system* $\mathbf{H}_A$ *is* 3. *By using Gröbner bases of* $\mathbf{H}_A$*, we have the following explicit difference Pfaffian system for* $F = {}^t(f,\ S_1 \bullet f,\ S_2 \bullet f)$*:*

$$S_1 F = A_1(x,s)F, \qquad S_2 F = A_2(x,s)F$$

Here $A_1(x,s)$ *and* $A_2(x,s)$ *are matrices of rational forms. We do not show the explicit expression of* A_1, A_2 *in this paper because they are more complicated. But at the point* $x = (1,1,0,1)$ *they are*

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3(6s_1-2s_2-3)(3s_1-s_2)}{31} & \frac{85s_1-31s_2-58}{31} & \frac{2((3s_1-s_2)^2-1)}{31} \\ \frac{3s_1-s_2}{2} & -\frac{3}{2} & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0 & 1 \\ \frac{3s_1-s_2}{2} & -\frac{3}{2} & 0 \\ -\frac{31s_1-13s_2}{4(3s_1-s_2+2)} & \frac{31}{4(3s_1-s_2+2)} & \frac{3(3s_1-s_2+1)}{2(3s_1-s_2+2)} \end{pmatrix} \end{aligned}$$

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Gröbner bases of Hilbert ideals

Takashi Wada
(Rikkyo University)

Hilbert ideal is an ideal generated by invariant polynomials under a finite group. For example, by virtue of a well-known result on symmetric polynomials, one can compute the reduced Gröbner basis of the Hilbert ideal $\mathcal{H}(S_n)$ of the symmetric group S_n with respect to a lexicographic order. On the other hand, the Hilbert ideal $\mathcal{H}(A_n)$ of the alternating group A_n is studied in [1] and [2] etc. In the present talk, we compute the reduced Gröbner basis of Hilbert ideal $\mathcal{H}(A_n)$ with respect to a lexicographic order explicitly. Moreover, the universal Gröbner basis (i.e., the union of all reduced Gröbner bases) of Hilbert ideals $\mathcal{H}(S_n)$ and $\mathcal{H}(A_n)$ are also constructed. This is a joint work with Hidefumi Ohsugi.

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ALGEBRAIC SHIFTING OF FINITE GRAPHS

Satoshi Murai
(Osaka University)

Let $G = ([n], E(G))$ be a graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with the edge set $E(G)$. A graph G is called *shifted* if for any edge $\{i, j\} \in E(G)$ and for any $i' \le i$ and $j' \le j$, one has $\{i', j'\} \in E(G)$. Algebraic shifting is an operation which associates a simplicial complex K with a shifted simplicial complex $\Delta(K)$. There are two types of algebraic shifting, i.e., *exterior algebraic shifting* $K \mapsto \Delta^e(K)$ and *symmetric algebraic shifting* $K \mapsto \Delta^s(K)$. We regard a graph as a 1-dimentional simplicial complex and study $\Delta^e(G)$ and $\Delta^s(G)$.

First, we study algebraic shifting of complete bipartite graphs. Let $K_{n,m}$ be the complete bipartite graph of size n, m. It is known that the exterior algebraic shifted graph of $K_{3,3}$ and the symmetric algebraic shifted graph of $K_{3,3}$ are different, that is, the form of exterior algebraic shifted graph of $K_{3,3}$ is $E(\Delta^e(K_{3,3})) = \{12, 13, 14, 15, 16, 23, 24, 25, 34\}$ and the form of symmetric algebraic shifted graph of $K_{3,3}$ is $E(\Delta^s(K_{3,3})) = \{12, 13, 14, 15, 16, 23, 24, 25, 26\}$. The form of $\Delta^e(K_{n,m})$ was computed by Kalai. We determine the form of $\Delta^s(K_{n,m})$ and find that $\Delta^e(K_{n,m}) \neq \Delta^s(K_{n,m})$ if and only if $K_{3,3} \subset K_{n,m}$.

Second, we study algebraic shifting of chordal graphs. A graph G is called *chordal* if every cycle of length > 3 has a chord. Aramova, Herzog and Hibi studied the relation between algebraic shifting and graded Betti numbers of the Stanley-Reisner ideal. In the case of graphs, graded Betti numbers of Stanley-Reisner ideal of $\Delta(G)$ determine the form of $\Delta(G)$. If G is chordal, Stanley-Reisner ideal of G and Stanley-Reisner ideal of $\Delta(G)$ have the same graded Betti numbers. This implies $\Delta^e(G) = \Delta^s(G)$. We will study some properties which do not change by algebraic shifting if G is chordal. Also, we give a combinatorial algorithm to compute algebraic shifting of chordal graphs.

An application of the Quillen-Suslin theorem to the construction of vector bundles

Hirotachi Abo
(Colorado State University)

A basic problem in algebraic geometry is the classification of projective varieties. A very interesting case is when the codimensions of projective varieties are small. One of the motivations to study such varieties is Hartshorne's conjecture. In 1974, he suggested that a nonsingular subvariety of projective space should be a complete intersection, when its codimension is small as compared with the dimension of the projective space. This conjecture is closely related to the question of whether there are vector bundles of small rank on the projective space, which are not direct sum of line bundles. So it is an important task to find such bundles. However this seems a very hard problem. Indeed, there is no such bundle known to exist on the projective space whose dimension is greater than or equal to 6. So a question is: "How can we attack this problem?"

There are several construction methods known. One of the powerful methods for constructing bundles is Kumar's construction. This construction is based on the well-known Serre's conjecture, which was solved by Quillen and Suslin independently. For a given vector bundle, the theorem of Quillen and Suslin guarantees us the existence of sections that generate the vector bundle on the complement of a hyperplane. The pair of the vector bundle and these sections corresponds to a vector bundle on this hyperplane. Kumar gave necessary and sufficient conditions for a vector bundle on a hyperplane of projective space to be obtained from a vector bundle on the projective space in this way. It turns out that there exists a correspondence between vector bundles on the n-dimensional projective space and vector bundles on the $(n-1)$-dimensional projective space satisfying certain conditions. By using this correspondence, Kumar established the existence of many rank two vector bundles on projective fourspace in positive characteristics.

A natural question is: "How can we compute a vector bundle on the n-dimensional projective space from the corresponding bundle on the $(n-1)$-dimensional projective space? The purpose of this talk is to develop a constructive method for this computation and to show by means of a couple of examples how this method works. This is joint work with Chris Peterson.

ALGORITHMIC RESOLUTION OF SINGULARITIES

Gerhard Pfister
(Universität Kaiserslautern)

An algorithmic proof of the famous theorem of Hironaka on the resolution of singularities is sketched. The algorithm and its implementation in SINGULAR is described. Applications (computation of the zeta-function, the spectrum etc.) are given.

EFFECTIVE COMPUTATION OF GRÖBNER BASES AND MARKOV BASES OF TORIC IDEALS

Raymond Hemmecke
(Otto-von-Guericke University Magdeburg)

Gröbner bases of toric ideals appear under the disguise of test sets in integer programming, while Markov bases (minimal generating sets) of toric ideals have applications for example in statistics. We will present these applications as well as a new algorithm to compute Markov bases and Gröbner bases of toric ideals. These algorithms have been implemented in the software package 4ti2, obtainable from www.4ti2.de. Our computational experiments show that 4ti2 is much faster than other available software packages when it comes to these computations. We conclude the talk with a short demonstration of 4ti2.

Listing the A-equivalence classes

Mutsumi Saito
(Hokkaido University)

Let A be a $d \times n$ integer matrix whose column vectors generate the lattice $\mathbb{Z}^d$, and let $D(R_A)$ be the ring of differential operators on the affine toric variety defined by A. We can define an equivalence relation in $\mathbb{C}^d$, called A-equivalence relation. Then the A-equivalence relation classifies A-hypergeometric systems and $\mathbb{Z}^d$-graded simple $D(R_A)$-modules (up to shift).

Let $\boldsymbol{a} \in \mathbb{C}^d$. We give an algorithm for listing all A-equivalence classes in $\boldsymbol{a} + \mathbb{Z}^d$.

DIFFERENTIAL OPERATORS AND INVARIANT THEORY

William N. Traves
(U. S. Naval Academy)

Building on my work with Mutsumi Saito on differential operators of toric varieties, I will describe several ways to compute the ring of differential operators on a ring of invariants. Invariant theory started out as a computational discipline, though in the early twentieth century these concerns took a back seat to structural questions about the ring of invariants. However, due to the new machinery – both mathematical and technological – at our disposal, and by the many applications of invariant theory, the computation of invariants is once again possible. I'll survey some of the classical approaches to compute invariants and explain how differential operators produce new invariants from old.

TERM ORDERS FOR THE IDEAL OF COMMUTING MATRICES

Feyja Hreinsdóttir
(University of Iceland)

Let $X = (x_{ij})$ and $Y = (y_{ij})$ be generic n by n matrices and $Z = XY - YX$. Let $S = k[x_{11}, \dots, x_{nn}, y_{11}, \dots, y_{nn}]$, where k is a field. The ideal generated by the entries of Z is the ideal of commuting matrices, denote it by I. It has been conjectured that $R = S/I$, the ring of commuting matrices is Cohen-Macaulay [1].

In [2] we found a term order that enabled us to compute the Gröbner basis in the case $n = 4$ and thus prove that R is CM. Surprisingly it turned out that a certain product order speeded up the computations enormously, in fact one could not at that time compute the Gröbner basis using the reverse lexicographic order. This term order resulted in a Gröbner basis with 294 elements in degrees 2–8. With a fast computer one can now compute the Gröbner basis with respect to reverse lexicographic ordering, it however takes hours and has more than 600 elements in degrees 2–13.

In recent years we have studied various improvements of my term order and the best one found so far gives a Gröbner basis with 51 element in degrees 2–6. This term order was found by exploiting a certain symmetry in the generators of the ideal. We pick which leading terms we want the generators to have in order to maximize cancellations in the S-polynomials computed in the first step of the Buchberger algorithm. We then order the variables in different blocks to achieve this for as many generators as possible.

In this talk I will describe the problem and how the term orders where found. If time permits I will also present some conjectures concerning the ring of commuting matrices that are given in [3]. Most of them were found by using the computer algebra systems Macaulay and Macaulay2.

References

[1] D. Bayer, M. Stillman and Ma. Stillman, Macanlay User Manual.

[2] F. Hreinsdottir, *A case Where Choosing a Product Order Makes the Calculations of a Gröbner Basis Much Faster*, J. Symbolic Computation **18** (1994), 373–378.

[3] F. Hreinsdottir, *Conjectures on the ring of commuting matrices*, accepted for publication in International Journal of Commutative Rings, Available at math.AC/0501465.

CREATION OPERATORS OF THREE WAY CONTINGENCY TABLES

Toshio Sakata and Ryuichi Sawae
(Kyushu University and Okayama Science University)

In this talk, we treat the sequential conditional test of three way contingency tables. That is, whenever we get a new sample we perform the conditional test and if at some stage we have a conclusive decision, then the test procedure is stopped. This may be especially useful in clinical statistical fields, because it is not ethical to continue the process until reaching to a predetermined sample size even if we already had a conclusive result. Here we confront with the problem how to construct Ω of each stage sequentially. Mathematically this is a problem of finding a surfective sets of maps

$$\mathcal{S} : \Omega(\alpha, \beta, \gamma) \to \Omega(\alpha + E_{pq}, \beta + E_{qr}, \gamma + E_{pr})$$

where $\Omega(\alpha, \beta, \gamma)$ denotes the set of contingency tables having the marginal two dimensional contingency tables, α, β and γ, and $\Omega(\alpha + E_{pq}, \beta + E_{qr}, \gamma + E_{pr})$ denotes the set of contingency tables having two dimensional contingency tables $\alpha + E_{pq}$, $\beta + E_{qr}$ and $\gamma + E_{pr}$ with E_{ij} being the matrix with 1 in (i, j) cell and 0 in other cells. In this talk we propose a candidate of surjective set of mappings based on a minimal Markov basis. Though any theoretical proof of surfectivity is not given for general three way tables, however, for some special cases, the proofs are given and for $3 \times 3 \times 3$ tables an experimental result which partially supports the conjecture is given.

Powers of ideals and Gröbner bases

Jürgen Herzog
(Universität Essen)

In this lecture we give a survey on the study of the homological properties of powers of a graded ideal I in a polynomial ring, using Gröbner basis techniques. There are two possible approaches to do this: the first approach is to compare for a given term order, the ideals $\mathrm{in}(I)^j$ and $\mathrm{in}(I^j)$. Obviously, the first ideal is always contained in the second, and they are rarely equal for all j. In a paper with Conca and Valla we give examples which show that the smallest exponent for which $\mathrm{in}(I)^j$ and $\mathrm{in}(I^j)$ differ, may be arbitrarily high, while on the other hand, we have $\mathrm{in}(I)^j = \mathrm{in}(I^j)$, if equality holds up to the relation type of $\mathrm{in}(I)$. Moreover, in a paper with Trung and Hoa we showed, that $\mathrm{in}(I^{sj}) \subset \mathrm{in}(I)^j$ for all j, where s is the codimension of I. The second approach to study powers of I is to consider the initial ideal of the defining ideal of the Rees ring of I. As shown in a joint paper with Cutkosky and Trung, this technique admits to prove that the regularity $\mathrm{reg}(I^j)$ is a linear function for large j, and to give a condition that guarantees that all powers of I have a linear resolution, as we demonstrated in a joint paper with Hibi and Zheng. It can also be used to show that all powers of certain monomial ideals have linear quotients. This allows to estimate the depth of I^j for all j, as pointed out in a recent paper with Hibi. We present classes of examples to demonstrate these results.

GRÖBNER BASES IN ALGEBRAIC STATISTICS

Henry P. Wynn
(London School of Economics)

In joint work with G. Pistone and E. Riccomagno (Torino), the application of Gröbner bases to statistics was developed. The first stage is to express an experimental design, or the support of a discrete probability distribution, as a zero dimensional algebraic variety, that is as a solution of a set of polynomial equations. This enables an exact polynomial interpolator to be built, via ideal quotienting, from the Gröbner basis. This is the starting point for fitting sub-models in statistics based on data at the design points. When, however, the variable is the logarithm of probabilities at the design points, we obtain by the same interpolation, a saturated exponential log-linear model. Sub-models are obtained in this case from independence models and conditional independence models. In that case the conditions induced on the probabilities are a toric variety. This can also be obtained by a power product, rather than a log-linear exponential formulation. G-basis methods can be used to understand the structure of the toric variety. This follows the work of B Sturmfels and co-workers. In this paper special attention is paid to non-standard design and incomplete tables.