On K3 surfaces defined over $\mathbf{Q}$ — Correction to the paper [1] in vol.43 (1994)

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1. In a private correspondence, Matthias Schuett has pointed out a mistake in my paper [1]. Namely it is in the proof of Theorem 1 which states that, for any K3 surface defined over the rational number field $\mathbf{Q}$, the $\mathbf{Q}$-Picard number cannot attain the maximum value $h^{1,1} = 20$. Moreover he has given an explicit example of a K3 surface with $\mathbf{Q}$-Picard number 20, by using a K3 surface with maximal singular fibre studied in my recent paper [2]. According to Klaus Hulek, he and Verrill have found other example of such, too.

Thus "Theorem 1" of [1] should be cancelled. Also "Corollary 3" of that paper should be cancelled which asserted that, for any elliptic curve $E/\mathbf{Q}(t)$ arising from an elliptic K3 surface, the Mordell-Weil rank of $E(\mathbf{Q}(t))$ is less than 18. As a consequence, "Question 2" (which motivated that paper) is still open: Does there exist an elliptic K3 surface defined over $\mathbf{Q}$ such that $E/\mathbf{Q}(t)$ has rank 18 (E being the generic fibre)?

2. What was wrong with the proof of Theorem 1 in [1]?

We argued as follows. Assume there is a K3 surface, say X, over $\mathbf{Q}$ with $\mathbf{Q}$-Picard number 20. The reduction $X(p)$ of X modulo p is a K3 surface over the prime field $\mathbf{F}_p$ such that $\rho(X(p)/\mathbf{F}_p) \geq 20$ for almost all p. Over the algebraic closure k_p of $\mathbf{F}_p$, the Picard number $\rho(X(p)) = \rho(X(p)/k_p)$ is known to be either 20 or 22.

The gap in the proof was precisely at the point where we overlooked the possibility for $\rho(X(p)/\mathbf{F}_p) = 21$, as pointed out by Schuett and Hulek. As a matter of fact, the proof there implies that $\rho(X(p)/\mathbf{F}_p) = 21$ occurs for an infinitely many prime p, instead of leading to a contradiction as claimed in the paper [1].

3. It should be remarked that the corresponding proof is correct, if we replace a K3 by an abelian surface. Namely we have

Proposition 1 *For any abelian surface A over $\mathbf{Q}$, the $\mathbf{Q}$-Picard number cannot attain the maximum value $h^{1,1}(A) = 4$, i.e. $\rho(A/\mathbf{Q}) < 4$.*

Proof Arguing as above, we claim that the possibility $\rho(A(p)/\mathbf{F}_p) = 5$ cannot occur. Indeed, it would imply in terms of the standard notation (cf. [1])

$$P_2(A(p)/\mathbf{F}_p, T) = (1 - pT)^5(1 + pT). \tag{1}$$

But this is impossible, because $H^2 \cong \Lambda^2(H^1)$ together with Frobenius action holds for the etale cohomology of $A(p)/k_p$. *q.e.d.*

Remark In the case of a K3 surface X as in **2**, on the other hand, we have

$$P_2(X(p)/\mathbf{F}_p, T) = (1 - pT)^{21}(1 + pT). \tag{2}$$

for infinitely many p. In particular, this gives an example of the sign change in the functional equation for $P_2(T) = P_2(X(p)/\mathbf{F}_p, T)$:

$$P_2(\frac{1}{p^2T}) = -\frac{1}{(pT)^{b_2}}P_2(T). \tag{3}$$

At any rate, it seems that the mistake is caused by very delicate difference between K3 surfaces and abelian surfaces even in the "singular" case ($\rho = h^{1,1}$) where their structure is very closely related (cf. [3]).

4. Let us mention the related work of Shafarevich [4] who has formulated a very interesting finiteness conjecture:

Conjecture 2 *For all K3 surfaces over a given number field, their Néron-Severi lattice* NS(X) *belong to a finite set of isomorphism classes.*

He has proven this for singular K3 (and abelian) surfaces in a stronger sense:

Theorem 3 *There is only a finite number of isomorphism classes of singular K3 surfaces over the complex number field which is defined over a number field of given degree.*

In particular, it follows that the number of isomorphism classes of singular K3 surfaces defined over $\mathbf{Q}$ is finite, and so is the number of those with $\mathbf{Q}$-Picard number 20. It will be an interesting question to determine these numbers.

Finally I would like to thank M. Schuett and K. Hulek for pointing out my error in their nice communications, and T. Katsura for useful discussion.

References

[1] Shioda, T.: On the rank of elliptic curves over $\mathbf{Q}(t)$ arising from K3 surfaces, Comment. Math. Univ. St. Pauli 43, 117–120 (1994).

[2] — : The elliptic $K3$ surfaces with a maximal singular fibre, C. R. Acad. Sci. Paris, Ser. I 336, 461–466 (2003).

[3] Inose, H., Shioda, T.: On singular K3 surfaces, in: Complex Analysis and Algebraic Geometry, Iwanami Shoten and Cambridge Univ. Press, 119–136 (1977).

[4] Shafarevich, I.R.: On the arithmetic of singular K3 surfaces, in: Algebra and Analysis, Proc. Chebotarev Conf. in 1994 /Kazan, Walter de Gruyter & Co., 103–108 (1996).

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